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Corrigendum

Corrigendum to and two open questions arising from the article "Approximation in rough native spaces by shifts of smooth kernels on spheres" [J. Approx. Theory 133 (2005) 269–283]

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Let S^d denote the d-sphere embedded in the $(d+1)$-dimensional Euclidean space R^{d+1}. For each $k = 0, 1, \ldots,$ let $\mathcal{H}_k^{(0)}$ be the linear space of homogeneous harmonic polynomials in $(d+1)$-variables of degree k, and for a nonnegative integer L, let

$$\mathcal{H}_L := \bigoplus_{k \leqslant L} \mathcal{H}_k^{(0)}.$$

The dimension of $\mathcal{H}_k^{(0)}$ is denoted by q_k. It is well-known that

$$q_0 = 1, \quad q_k = \frac{(2k+d-1)\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d)}.$$

For each $k = 0, 1, \ldots,$ let $\{Y_{k,j} : j = 1, \ldots, q_k\}$ be an orthonormal basis of $\mathcal{H}_k^{(0)}$ with respect to the standard inner product

$$\langle p, q \rangle = \int_{S^d} p(x)q(x)\, d\mu(x), \quad p, q \in \mathcal{H}_k^{(0)},$$

where $d\mu$ denotes the rotational invariant measure on S^d whose total mass is denoted by ω_d, i.e., $\omega_d = \int_{S^d} d\mu(x)$. The set

$$\{Y_{k,\mu} : \mu = 1, \ldots, q_k,\ k = 0, 1, \ldots,\}$$

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forms an orthonormal basis for $L^2(S^d)$. For each fixed k, $\mathcal{H}_k^{(0)}$ is also the eigen-space of the Laplace–Betrami operator on S^d corresponding the eigenvalue $-\lambda_k$, where $\lambda_k := k(k+d-1)$, $k = 0, 1, 2, \ldots$.

In [LS], we claimed the following result in Lemma 4.1:

Let M be a (positive) multiplier operator defined on $\mathcal{H}_L$ (embedded in $C(S^d)$) by

$$M(p) = \sum_{k=0}^{L} \sum_{\mu=1}^{q_k} m_{k,\mu} a_{k,\mu} Y_{k,\mu},$$

where

$$p = \sum_{k=0}^{L} \sum_{\mu=1}^{q_k} a_{k,\mu} Y_{k,\mu},$$

and $m_{k,\mu}$ are positive real numbers. Then

$$\|M(p)\| \leqslant C \left(\max_{k,\mu} m_{k,\mu} \right) \|p\|,$$

where C is a constant independent of p and L, and $\|\cdot\|$ denotes the standard norm in $C(S^d)$.

We also stated that this result is a special case of a general Bernstein-type inequality established by Ditzian [D, Theorem 3.2]. Recently, Ditzian has kindly pointed out that the result as claimed above, without any restriction on the multipliers $m_{k,\mu}$, does not follow from his Theorem 3.2. He has also indicated that Lemma 4.1, in its most general form, may be too good to be true. We now state and prove a weaker version of Lemma 4.1.

Lemma 4.1*. *Let α be a nonnegative real number, and let M be the multiplier operator defined on $\mathcal{H}_L$ (embedded in $C(S^d)$) by*

$$M(p) = \sum_{k=0}^{L} (\lambda_k)^{\alpha} \sum_{\mu=1}^{q_k} a_{k,\mu} Y_{k,\mu},$$

where

$$p = \sum_{k=0}^{L} \sum_{\mu=1}^{q_k} a_{k,\mu} Y_{k,\mu}.$$

Then

$$\|M(p)\| \leqslant C (\lambda_L)^{\alpha} \|p\|,$$

where C is a constant independent of p and L.

Proof. This is a special case of Theorem 3.2 in [D]. In fact, in Theorem 3.2 in [D], one takes the differential operator $P(D)$ to be the Laplace–Betrami operator on S^d whose eigen-space corresponding to the eigenvalue, $-\lambda_k$, is $\mathcal{H}_k^{(0)}$. As pointed out in Section 9 in [D] the Cesàro summability of order l is valid for $l > \frac{d-1}{2}$ and for all $L_p(S^d)$, $1 \leqslant p \leqslant \infty$. Note that in [D] the crucial index is recorded as $\frac{d-2}{2}$. The difference can be reconciled by observing that Ditzian used

S^{d-1} as the domain while we have S^d. Simply put, the d in [LS] plays the role of $d-1$ in [D]. Besides the references [BC,SW] cited in [D], the validity of the lth order Cesàro summability for $l > \frac{d-1}{2}$ was alluded to by Kogbetliantz [K] in 1924. Some even attribute the summability result to Gegenbauer in as early as 1880 [A]. □

The weaker Lemma 4.1* also has a ramification to the error estimates established in Section 4 in [LS]. These error estimates now hold for the following types of Xu–Cheney [XC] kernels as we defined in [LS, Section 2]:

$$\phi(xy) = a_0 + \sum_{k=1}^{\infty}(\lambda_k)^{-\alpha}\sum_{\mu=1}^{q_k} Y_{k,\mu}(x)Y_{k,\mu}(y), \tag{1}$$

$$\psi(xy) = b_0 + \sum_{k=1}^{\infty}(\lambda_k)^{-\beta}\sum_{\mu=1}^{q_k} Y_{k,\mu}(x)Y_{k,\mu}(y). \tag{2}$$

Here α, β ($\alpha \geqslant \beta > d/2$) are two nonnegative real numbers, and a_0, b_0 are two positive real numbers. We refer to ϕ as the "smoother kernel", and ψ the "rough kernel". Let Ξ be a nonempty finite subset of S^d. We embed the $|\Xi|$-dimensional space: span$\{x \mapsto \phi(xy) : y \in \Xi\}$ in the native space $\mathcal{N}_\psi$, and consider the best approximation of this finite-dimensional space to an arbitrary function f from $\mathcal{N}_\psi$. Various error estimates were established for $\| f - s_\phi(f)\|$ in [LS], where $s_\phi(f)$ denotes the best approximant of f in $\mathcal{N}_\psi$. Narcowich and Ward [NW] have showed that the native space $\mathcal{N}_\psi$ is isometric to the Sobolev space $H_\beta(S^d)$ defined by

$$H_\beta(S^d) := \left\{ f \in \mathcal{D}'(S^d) : \|f\|^2_{H_\beta(S^d)} := \sum_{k,\mu}(\lambda_k + 1)^\beta |\hat{f}_{k,\mu}|^2 < \infty \right\}.$$

Here $\mathcal{D}'(S^d)$ denotes the space of all tempered distributions on S^d.

We now replace Theorem 4.3 by the following Theorem 4.3*.

Theorem 4.3*. *Let ϕ and ψ be the Xu–Cheney kernels as written in Eqs.* (1) *and* (2), *and let L be a natural number. Assume that a nonempty finite subset Ξ of S^d satisfies $h(\Xi) \leqslant 1/(2L)$. Then for each fixed $x \in S^d$, we have*

$$\|\psi_x - s_\phi[\psi_x]\|_{\mathcal{N}_\psi} \leqslant C \left(\sum_{k>L}^{\infty} b_k d_k \right)^{1/2}.$$

Here the constant C is independent of f and x.

The highlight of the results in Section 4 [LS] is that $\|f - s_\phi(f)\|$ enjoys the same order of error estimate as $\|f - s_\psi(f)\|$. Here $s_\psi(f)$ denotes the best approximant of f from the $|\Xi|$-dimensional space: span$\{x \mapsto \psi(xy) : y \in \Xi\}$ in $\mathcal{N}_\psi$.

The multiquadrics kernel $(1 + xy)^{1/2}$ and its generalizations (for example, the inverse multiquadrics: $(1 + xy)^{-1/2}$) also provide efficient approximation tools on spheres. Their expansions in terms of spherical harmonics have coefficients that decay exponentially; see [H]. Therefore Theorem 4.3* does not apply to these kernels.

Let Ξ be a nonempty finite subset of S^d, and denote the $|\Xi|$-dimensional space: span$\{x \mapsto (1+xy)^{1/2} : y \in \Xi\}$, by M_Ξ. Embed M_Ξ in the native space $\mathcal{N}_\psi$, and, for an arbitrary function f from $\mathcal{N}_\psi$, consider the best approximation of f from M_Ξ. The following two questions then naturally arise:

Open Question I. Let M be a multiplier operator defined on $\mathcal{H}_L$ (embedded in $C(S^d)$) by

$$M(p) = \sum_{k=0}^{L} m_k \sum_{\mu=1}^{q_k} a_{k,\mu} Y_{k,\mu},$$

where

$$p = \sum_{k=0}^{L} \sum_{\mu=1}^{q_k} a_{k,\mu} Y_{k,\mu},$$

the multipliers $\{m_k\}$ satisfy $0 < m_0 < m_1 < m_2 < \cdots$, and m_k grows exponentially with k. Is there a constant C, independent of p and L, such that

$$\|M(p)\| \leqslant C m_L \|p\|,$$

where $\|\cdot\|$ denotes the standard norm in $C(S^d)$.

Open Question II. For an arbitrary function f in $\mathcal{N}_\psi$, is the order of approximation of f from M_Ξ comparable to that from span$\{x \mapsto \psi(xy) : y \in \Xi\}$?

Remark. If the answer to Question I is affirmative, then one can use the method in [LS] to give an affirmative answer to Question II. On the other hand, it is quite possible that one can answer Question II independent of Question I.

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